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The structure of the invariants of perfect Lie algebras

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Abstract
Upper bounds for the number $\mathcal{N}(\mathfrak{g})$ of Casimir operators of perfect Lie algebras $\mathfrak{g}$ with nontrivial Levi decomposition are obtained, and in particular the existence of nontrivial invariants is proved. It is shown that for high-ranked representations R the Casimir operators of the semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ of a semisimple Lie algebra $\mathfrak{s}$ and an Abelian Lie algebra $(\deg R)L_1$ of dimension equal to the degree of R are completely determined by the representation R, which also allows the analysis of the invariants of subalgebras which extend to operators of the total algebra. In particular, for the adjoint representation of a semisimple Lie algebra the Casimir operators of $\mathfrak{s}\overrightarrow{\oplus}_{ad(\mathfrak{s})}(\dim\mathfrak{s})L_1$ can be explicitly constructed from the Casimir operators of the Levi part $\mathfrak{s}$.

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1. Introduction

The problem of characterizing the number and form of invariants of the coadjoint representation of Lie algebras has gained importance not only in representation theory, in order to label and characterize representations, but also in physical applications, where the eigenvalues of the corresponding invariants, in particular the Casimir operators, provide quantum numbers describing the different states of a system, as well as other quantities characterizing specific properties of the system [1]. For semisimple Lie algebras the problem was entirely solved in 1950s [2, 3], partially motivated by the application of group theory to spectroscopy and the theory of branching rules necessary to classify the electron l^n-configurations in total angular momentum and spin couplings (LS or Russel–Saunders couplings) and the determination of energy matrices [4]. For the nonsemisimple case only very few results exist, mainly isolated types of algebras which have been analysed because of their physical interest (see [5] and references therein). This is, for example, the case for the inhomogeneous Lie algebras $\mathfrak{sa}(n,\mathbb{R})$, which appear in gauge theories of gravity [6]. However, a general theory of invariants

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concerning these algebras does not exist, since there is no analogue of the root method for the nonsemisimple case. Of special interest are those algebras whose Levi decomposition is nontrivial, since most of the symmetry groups appearing in physics are a semidirect product of some semisimple group and an Abelian or solvable group (see for instance the kinematical Lie algebras [7]). However, here the difficulty in obtaining the invariants is sometimes formidable, since they depend simultaneously on the radical (the maximal solvable ideal) and the Levi part, specifically on the representation describing the semidirect product. It has been shown that the Levi decomposition does not simplify the determination of the invariants [8], so that new methods must be developed in this case in order to obtain some general criteria.

In this paper we concentrate on the invariants of perfect Lie algebras, i.e., of Lie algebras satisfying the equality $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and which are not semisimple. In view of the Levi decomposition, this is equivalent to ensuring that the radical is not reduced to zero. We first give an answer to a question that has already appeared in the literature, namely if perfect Lie algebras must necessarily have nontrivial invariants [9]. It was shown long ago that, if invariants exist, they can be chosen as Casimir operators [10, 11]. Using classical theory we show that the perfectness establishes the existence of invariants. This will be a consequence of the radical's structure. In contrast to the classical algebras, the number of independent invariants cannot, however, be deduced from the rank of the semisimple part. We will establish some approximation formulae for the number of invariants, which are indeed upper bounds. These formulae can be obtained with complete independence from the structure of the radical, and are obtained in terms of the Levi part and the representation R describing the semidirect product. The case of the Lie algebras $\mathfrak{s} \overrightarrow{\oplus}_{ad\mathfrak{s}} (\dim \mathfrak{s}) L_1$ is of special interest, since the number of invariants is completely determined by the rank of the semisimple part $\mathfrak{s}$ and they can be explicitly constructed from the Casimir operators of $\mathfrak{s}$. It will follow, in particular, that these algebras admit always even number or quadratic Casimir operators. Moreover, for sufficiently high-dimensional representations, these formulae will give the exact number of independent invariants of the algebra. This is of interest for tensor products, since from certain dimensions onwards this implies that the Casimir operators of the algebra are completely determined by the representation. This question also leads to the analysis of extending invariants of certain subalgebras to invariants of the total algebra. For the case of Abelian radicals (the only case which can be dealt with in full generality) this will allow us to establish some lower bounds for the number of invariants. As a consequence, the number of Casimir operators that depend on the whole representation (and not merely on some irreducible component) can be predicted, which constitutes an important fact in some applications like the analysis of missing label operators [5].

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ considered in this work is indecomposable and is defined over the field $\mathbb{R}$ of real numbers. We convene that nonwritten brackets are either zero or obtained by antisymmetry. We also use the Einstein summation convention. Abelian Lie algebras of dimension m will be denoted by mL_1.

2. Invariants of Lie algebras

We discuss briefly the method used to find these invariants, and in particular the Casimir operators of an algebra [12]. If $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$ and $\left\{C_{ij}^k\right\}$ the structure constants over this basis, we can represent $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ by the differential operators,

$$\hat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j} \qquad (1)$$

where $[X_i, X_j] = C_{ij}^k X_k\,(1 \leqslant i < j \leqslant n)$. The operators $\hat{X}_i$ satisfy the brackets $[\hat{X}_i, \hat{X}_j] = C_{ij}^k \hat{X}_k$ and therefore constitute a representation of $\mathfrak{g}$. An analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system:

$$\{\hat{X}_i F = 0, 1 \leqslant i \leqslant n\}. \tag{2}$$

Polynomial solutions of the system correspond to classical Casimir invariants, after symmetrization. The system (2) can also have solutions which are not polynomials (e.g. rational functions or harmonics), which leads naturally to enlarge the concept of invariant to 'generalized Casimir invariants'. These solutions also have a fixed value on irreducible representations of $\mathfrak{g}$. If (2) has no solutions at all (as happens, for example, for the two-dimensional affine algebra $\mathfrak{r}_2$) we say that the invariants of the coadjoint representation ad^* are trivial. In particular, an adaptation of a well-known result of partial differential equations allows us to determine the cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions of the system in terms of the brackets of $\mathfrak{g}$ over a given basis,

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \sup_{x_1,\ldots,x_n} \left\{\operatorname{rank}\left(C_{ij}^k x_k\right)_{1\leqslant i<j\leqslant \dim \mathfrak{g}}\right\} \tag{3}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, \ldots, X_n\}$ [13].

As an example, let us consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{2D_1} 6L_1$, where D_j denotes the $(2j+1)$-dimensional irreducible representation of maximal weight $\lambda = 2j$. The commutator matrix $A(\mathfrak{g})$ with respect to the ordered basis $\{X_1, \ldots, X_9\}$ is

$$A(\mathfrak{g}) = \left(\begin{array}{ccc|ccc|ccc}
0 & 2x_2 & -2x_3 & 2x_4 & 0 & -2x_6 & 2x_7 & 0 & -2x_9 \\
-2x_2 & 0 & x_1 & 0 & 2x_4 & x_5 & 0 & 2x_7 & x_8 \\
2x_3 & -x_1 & 0 & x_5 & 2x_6 & 0 & x_8 & 2x_9 & 0 \\
\hline
-2x_4 & 0 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2x_4 & -2x_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
2x_6 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-2x_7 & 0 & -x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2x_7 & -2x_9 & 0 & 0 & 0 & 0 & 0 & 0 \\
2x_9 & -x_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{4}$$

It is immediate that the rank of this matrix is 6, thus there are three independent invariants. Taking into account the subdivision of the matrix above, and applying the method introduced in [8] to reduce the corresponding system (2), we easily obtain a fundamental set of invariants for this algebra, formed by the Casimir operators $I_1 = 4x_4x_6 - x_5^2$, $I_2 = 4x_7x_9 - x_8^2$ and $I_3 = x_1x_5 - 2x_3x_4 - x_2x_6$. We observe that no fundamental system of invariants formed by functions which are independent of the variables $\{x_1, x_2, x_3\}$ can be obtained.

Invariants of Lie algebras for the coadjoint representation have been determined only in low dimensions, due to the nonexistence of complete classifications in dimensions $n \geqslant 7$ (for nonsemisimple algebras). Lie algebras with nontrivial Levi part have been classified up to dimension 9 [14], and partially in dimension 10 [15]. Invariants of Lie algebras having a rank 1 Levi subalgebra have been analysed in [8], where some general formulae on their number and structure were given. Further, there exist various results in low dimensions and for special types of solvable Lie algebras [5, 9, 16, 17]. Here we will concentrate on a special type of Lie algebras, called perfect, and which includes in particular semisimple Lie algebras.

Definition 1. *A Lie algebra* $\mathfrak{g}$ *is called perfect if* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Nonsemisimple perfect Lie algebras arise when contracting semisimple Lie algebras. Thus, for example, the Poincaré Lie algebra in (3 + 1) dimensions is a perfect Lie algebra obtained from a contraction of the de Sitter Lie algebra $\mathfrak{so}(4,1)$ [7].

3. Existence of Casimir operators for perfect Lie algebras

Although, in general, formula (3) constitutes only an upper bound for the number of independent Casimir operators [11], for perfect Lie algebras $\mathfrak{g}$ we have that $\mathcal{N}_C(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$, where $\mathcal{N}_C(\mathfrak{g})$ denotes the number of independent Casimir operators [11]. This is due to the fact that perfect Lie algebras are algebraic [10].

This result does not however solve the question of existence, since a perfect Lie algebra may have no invariants at all. Consider for example the special affine Lie algebras $\mathfrak{sa}(n,\mathbb{R})$, which are known to have only one invariant, independent of the dimension [18]. It could therefore be expected that there exists some semisimple Lie algebra $\mathfrak{s}$, some representation R of it and a solvable Lie algebra $\mathfrak{r}$ such that $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ is perfect and has no invariants. In this section we show that this cannot occur, that is, a perfect algebra always admits nontrivial invariants, which are moreover classical Casimir operators. The key to deriving this is a technical result due to Dixmier in his study of universal enveloping algebras $\mathfrak{U}(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ [19]:

Theorem 1. *Let $\mathfrak{g} \neq 0$ be a finite-dimensional Lie algebra. Let*

$$S := \{u \in \mathfrak{U}(\mathfrak{g}) \mid [u, x] = \lambda(x)u, \quad \forall x \in \mathfrak{g}\}$$

be the semicentre of $\mathfrak{U}(\mathfrak{g})$. Then S does not reduce to the scalars $\mathbb{R}$.

The semicentre S is indeed a commutative subalgebra of $\mathfrak{U}(\mathfrak{g})$ that contains the centre $Z(\mathfrak{U}(\mathfrak{g}))$. If now $\mathfrak{g}$ is a Lie algebra whose radical (i.e. maximal solvable ideal) is nilpotent, then it can be shown, using what Dixmier called 'distinguished linear forms on $\mathfrak{g}$', that $S = Z(\mathfrak{U}(\mathfrak{g}))$ [19]. This implies that any rational solution to the system (2) can be reduced to a polynomial solution. Thus in the case of Lie algebras with a nilpotent radical we can always find Casimir operators. Semisimple Lie algebras are a special case of this corresponding to a vanishing radical. Theorem 1 reduces the existence problem of Casimir operators on perfect algebras to the following:

Proposition 1. *Let $\mathfrak{g}$ be a perfect Lie algebra with Levi decomposition $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$. Then its radical $\mathfrak{r}$ is a nilpotent Lie algebra.*

Proof. Consider the ideal $J = [\mathfrak{g},\mathfrak{g}] \cap \mathfrak{r}$. Since $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$, we conclude that $J = \mathfrak{r}$. In fact, from the Levi decomposition of $\mathfrak{g}$ we obtain

$$[\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}, \mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}] \subset \mathfrak{s}\overrightarrow{\oplus}_R[\mathfrak{g},\mathfrak{r}] \tag{5}$$

and since $\mathfrak{g}$ is perfect $\mathfrak{r} = [\mathfrak{g},\mathfrak{r}] = J$. If now (V,ρ) is an irreducible $\mathfrak{g}$-module, by Lie's theorem there exists a common eigenvector $v_0 \neq 0$ for all elements of the radical $\mathfrak{r}$. If we define

$$W_f = \{v \in V \mid \rho(X)(v) := X \cdot v = f(X) \cdot v \quad \forall X \in \mathfrak{r}\}$$

the nonnullity of v_0 implies the existence of a linear form $f_0 \in \mathfrak{g}^*$ such that $W_{f_0} \neq 0$. Now this space is stable by the action of $\mathfrak{g}$, and by irreducibility of the representation it follows that $V = W_{f_0}$. Therefore the action of the radical on the module is by scalar transformations, which implies that $[\mathfrak{g},\mathfrak{r}] \cdot V = 0$. This proves that $\rho(X)$ is nilpotent for the elements $X \in \mathfrak{r}$, from which the nilpotence of $\mathfrak{r}$ follows from application of Engel's theorem. □

Corollary 1. *A perfect Lie algebra* $\mathfrak{g}$ *satisfies* $\mathcal{N}(\mathfrak{g}) > 0$. *Moreover,* $\mathfrak{g}$ *admits a fundamental set of invariants* $\mathcal{F}$ *formed by Casimir operators.*

Since the radical must be nilpotent, from proposition 1 we deduce the existence of nontrivial Casimir operators. By the formula $\mathcal{N}_C(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ proved in [11] we can find a maximal set of independent invariants formed by Casimir invariants. This result, which follows naturally from the classical theory, allows us to reinterpret the existence of the quadratic Casimir operator for semisimple Lie algebras (and the other operators of higher degree). Although it seemed that the existence of the invariants is a consequence of the nondegeneracy of the Killing form, the preceding results point out that the reason for their existence lies principally in the fact that the algebra is perfect. The effect of the semisimplicity will have consequences on the number of independent invariants (given by the rank), their minimal degree (as is the case of the quadratic operator) and on the integrity properties of the fundamental system formed by the Casimir operators [3]. For a general perfect Lie algebra these properties no longer hold, as can easily be extracted from the special affine Lie algebras $\mathfrak{sa}(n, \mathbb{R})$, which have only one Casimir operator independent of the rank of the simple part [18]. Another interesting fact is the independence of the particular structure of the radical $\mathfrak{r}$ as nilpotent Lie algebra. This also shows that the characterization problem of invariants of Lie algebras reduces to the analysis of the Casimir operators of classical algebras $\mathfrak{s}$, their representations and the solvable non-nilpotent Lie algebras admitting an $\mathfrak{s}$-module structure [8, 20].

A particular case follows immediately from proposition 1, namely the twisted product of a semisimple Lie algebra $\mathfrak{r}$ with an Abelian algebra determined by an irreducible representation of $\mathfrak{s}$:

Corollary 2. *For any semisimple Lie algebra* $\mathfrak{s}$ *and any irreducible representation R the Lie algebra* $\mathfrak{s} \overrightarrow{\oplus}_R (\deg R) L_1$ *admits a fundamental set of invariants formed by Casimir operators.*

This is a stronger version of theorem 2 in [8], since the existence of Casimir invariants is ensured for any rank. The second assertion follows at once from the special case of the Beltrametti–Blasi formula (3) considered in [11]. In particular, we deduce an interesting consequence concerning the Lie algebras having no invariants [8, 16, 20]:

Corollary 3. *If* $\mathcal{N}(\mathfrak{g}) = 0$*, then the factor algebra* $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ *does not reduce to zero.*

The interesting consequence of this for classification purposes is that a perfect Lie algebra can never contract (in the Inönü–Wigner sense) to an algebra admitting only trivial invariants for the coadjoint representation. This fact has a notorious significance in the analysis of the orbits of Lie algebras and their topology, as well as in the analysis of rigidity properties on solvable Lie algebras [21, 22].

A problem that arises naturally from this is to obtain some estimation of the number of independent Casimir operators of a perfect Lie algebra.

4. Upper bound for Casimir operators

In the preceding section we have seen that a perfect Lie algebra always admits nontrivial invariants, due to the nilpotency of the radical, and that following proposition 1 we can indeed find a fundamental system of invariants formed by Casimir operators. However, for high-dimensional representations it is usually difficult to determine the number $\mathcal{N}(\mathfrak{g})$ of independent invariants (see, e.g., perfect algebras with an exceptional Lie algebra as Levi part), up to some particular cases. It is therefore of interest to obtain some bounds for $\mathcal{N}(\mathfrak{g})$. The first approximation, which is rather unsatisfactory, is based only on the well-known properties of semisimple Lie algebras:

Lemma 1. *For a perfect Lie algebra* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ *the following inequality holds,*

$$\mathcal{N}(\mathfrak{g}) < \dim \mathfrak{r} + \dim \mathfrak{h} \tag{6}$$

where $\mathfrak{h}$ *is the Cartan subalgebra of* $\mathfrak{s}$.

Proof. This is a direct consequence of the fact that the number of independent Casimir operators of the Levi part is given by its rank [3]. Therefore,

$$\begin{aligned}\mathcal{N}(\mathfrak{g}) &= \dim \mathfrak{g} - \operatorname{rank} \mathrm{A}(\mathfrak{g}) = \dim \mathfrak{s} + \dim \mathfrak{r} - \operatorname{rank} \mathrm{A}(\mathfrak{g}) \\ &\leqslant \dim \mathfrak{s} + \dim \mathfrak{r} - \operatorname{rank} \mathrm{A}(\mathfrak{s}) = \dim \mathfrak{r} + \dim \mathfrak{h}.\end{aligned} \tag{7}$$

The strict character of the inequality follows at once from the fact that the rest of the commutator matrix $A(\mathfrak{g})$ (once $A(\mathfrak{s})$ has been extracted) cannot have rank zero. □

This first approximation will, however, be quite inexact, even in low dimensions. Consider for example the Lie algebra $\mathfrak{so}(3) \overrightarrow{\oplus}_{2ad\mathfrak{so}(3)} 6L_1$. This nine-dimensional algebra has three independent invariants, but formula (6) would only tell that $\mathcal{N} \leqslant 7$. For other Lie algebras, such as the exceptional algebras and their representations, the bound given by (6) is scarcely of use. Observe that this bound takes no care of the representation R used to describe the semidirect sum. One can therefore try to improve the upper bound of $\mathcal{N}(\mathfrak{g})$ taking into account the representation R of the Levi part acting on the radical. In order to be applicable, this bound should be independent of the particular structure of the (nilpotent) radical (the general case will follow from using this contraction). To this extent we introduce some additional notation: let $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ be a perfect algebra, $\{X_1, \ldots, X_n\}$ a basis of $\mathfrak{s}$, and $\{X_{1+n}, \ldots, X_{n+m}\}$ a basis of $\mathfrak{r}$. We define the matrix

$$\rho_R(\mathfrak{s}) := \left(-C_{i,n+j}^{n+k} x_{k+n}\right)_{1\leqslant i\leqslant n, 1\leqslant j,k\leqslant m} \tag{8}$$

that is, the matrix $\rho_R(\mathfrak{s})$ comprises the action of the Levi part on $\mathfrak{r}$ with respect to the representation R. In what follows, and unless otherwise stated, we will implicitly assume that R does not contain copies of the trivial representation D_0 of $\mathfrak{s}$.

Proposition 2. *For the Lie algebra* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ *the following inequality holds:*

$$\mathcal{N}(\mathfrak{g}) \leqslant \dim \mathfrak{r} + \dim \mathfrak{h} - \operatorname{rank} \rho_R(\mathfrak{s}). \tag{9}$$

Proof. Let $A(\mathfrak{g})$ be the commutator matrix of $\mathfrak{g}$. This matrix can be seen as a block matrix considering the following subdivision:

$$A(\mathfrak{g}) := \left(\begin{array}{c|c} A(\mathfrak{s}) & \rho_R(\mathfrak{s}) \\ \hline -\rho_R(\mathfrak{s})^t & A(\mathfrak{r}) \end{array}\right). \tag{10}$$

Then we have $\operatorname{rank} \mathrm{A}(\mathfrak{g}) \geqslant \operatorname{rank} \mathrm{A}(\mathfrak{s}) + \operatorname{rank} \rho_R(\mathfrak{s})$ (since $\operatorname{rank} \mathrm{A}(\mathfrak{s}) < \dim \mathfrak{s}$ and the apportation of $A(\mathfrak{r})$ is being ignored), so that

$$\mathcal{N}(\mathfrak{g}) \leqslant \dim \mathfrak{r} + \dim \mathfrak{s} - \operatorname{rank} \mathrm{A}(\mathfrak{s}) - \operatorname{rank} \rho_R(\mathfrak{s}) = \dim \mathfrak{r} + \dim \mathfrak{h} - \operatorname{rank} \rho_R(\mathfrak{s}). \tag{11}$$

□

In general, if the degree $\deg R$ of the representation is very high in comparison to the dimension of $\mathfrak{s}$, we will have that $\operatorname{rank} \rho_R(\mathfrak{s}) = \dim \mathfrak{s}$, so that in this case (11) can be reduced to

$$\mathcal{N}(\mathfrak{g}) \leqslant \dim \mathfrak{r} + \dim \mathfrak{h} - \dim \mathfrak{s}. \tag{12}$$

Table 1. Evaluation of (15) for $\mathfrak{s} = \mathfrak{su}(3)$.

R	dim $\mathfrak{g}$	rank A($\mathfrak{g}$)	$\mathcal{N}(\mathfrak{g})$	Upper bound (15)	$\mathcal{M}(\mathfrak{g})$
3	11	10	1	2	0
$3 \oplus 3$	14	12	2	2	0
$3 \oplus \bar{3}$	14	12	2	3	1
6	14	12	2	3	1
8	16	12	4	4	2
$6 \oplus 3$	17	16	1	3	1
$6 \oplus \bar{3}$	17	14	3	3	1
$3 \oplus 3 \oplus 3$	17	16	1	1	1
$3 \oplus 3 \oplus \bar{3}$	17	14	3	3	2
10	18	16	2	4	2
$6 \oplus 6$	20	16	4	6	4
3^4	20	16	4	4	4
3^5	23	16	7	9	7

Indeed (11) (or (12) when applicable) will be the best upper bound that can be obtained without having more information about the structure of the radical. One could be tempted to use the decomposition of R into irreducible representations, and to develop a similar argument as in the preceding proposition. However, a further subdivision will soon lead to false results, as can be seen for the following example. Let $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ and consider the tensor product representation $R = D_2 \otimes D_1$. We have $R = D_3 \oplus D_2 \oplus D_1$, and the representation is of dimension 15. If we assume that $\mathfrak{r} = 15L_1$, then $\mathfrak{s} \overrightarrow{\oplus}_R 15L_1$ has 12 Casimir operators. The upper bound above provides the approximation

$$\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}) \leqslant 15 + 1 - \mathrm{rank}\, \rho_R(\mathfrak{s}) = 13 \tag{13}$$

and by parity of the dimension the value is at most 12. Any further reduction that considers the decomposition of R into irreducible components would give a value which is lower than the number of Casimir operators of the algebra. Although the bound (11) cannot be refined using the decomposition of R, we can improve it by considering the parity, as in the preceding example. If we define

$$\epsilon(\mathfrak{g}) := \begin{cases} 1 & \text{if} \quad \mathrm{rank}\, \rho_R(\mathfrak{s}) \equiv 1 \pmod 2 \\ 0 & \text{if} \quad \mathrm{rank}\, \rho_R(\mathfrak{s}) \equiv 0 \pmod 2 \end{cases} \tag{14}$$

then formula (11) is rewritten as

$$\mathcal{N}(\mathfrak{g}) \leqslant \dim \mathfrak{r} + \dim \mathfrak{h} - \mathrm{rank}\, \rho_R(\mathfrak{s}) - \epsilon(\mathfrak{g}). \tag{15}$$

Formula (15) provides a good approximation to the value of $\mathcal{N}(\mathfrak{g})$. Table 1 shows the values obtained for some representations of $\mathfrak{su}(3)$ up to degree 12. Here m denotes an m-dimensional irreducible representation, while $\bar{m}$ denotes dual representations.

We further note that for the tabulated representations of $\mathfrak{su}(3)$ we have

$$\mathcal{N}(\mathfrak{su}(3) \overrightarrow{\oplus}_R (\deg R) L_1) = \mathcal{N}(\mathfrak{su}(3) \overrightarrow{\oplus}_{\bar{R}} (\deg \bar{R}) L_1) \tag{16}$$

where $\bar{R}$ denotes the dual of the representation R.

Formula (15) contains more information than a mere upper bound for the number of independent invariants, since it allows us to generalize some of the results of [8] to higher rank algebras. The case of an Abelian radical suffices to establish an upper bound for the general case of arbitrary nilpotent radicals, as shown in the following:

Proposition 3. *Let* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ *be a perfect Lie algebra. Then*

$$\mathcal{N}(\mathfrak{g}) \leqslant \dim \mathfrak{r} + \dim \mathfrak{h} - \operatorname{rank} \rho_R(\mathfrak{s}) - \varepsilon(\mathfrak{g}).$$

Proof. Let $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}\}$ be a basis of $\mathfrak{g}$ such that $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{s}$ and $\{X_{n+1}, \ldots, X_{n+m}\}$ is a basis of $\mathfrak{r}$. Let $\left\{C_{ij}^k\right\}_{1\leqslant i,j,k\leqslant n+m}$ be the structure constants of $\mathfrak{g}$ over this basis. If we consider the change of basis defined by

$$X_i' := X_i \qquad 1 \leqslant i \leqslant n \qquad X_{n+j}' := \frac{1}{n} X_{n+j} \qquad 1 \leqslant j \leqslant m \tag{17}$$

then, with respect to this change, the brackets of the algebra change to

$$\begin{aligned} &[X_i', X_j'] = [X_i, X_j] && 1 \leqslant i, j \leqslant n \\ &[X_i', X_{n+j}'] = \frac{1}{n}[X_i, X_{n+j}] && 1 \leqslant i \leqslant n \quad 1 \leqslant j \leqslant m \\ &[X_{n+i}', X_{n+j}'] = \frac{1}{n^2}[X_{n+i}, X_{n+j}] && 1 \leqslant i, j \leqslant m. \end{aligned} \tag{18}$$

Since $[X_i', X_{n+j}'] = \frac{1}{n} C_{i,j+n}^{n+k} X_{n+k} = C_{i,j+n}^{n+k} X_{n+k}'$, (17) shows that the Levi part $\mathfrak{s}$ and the representation R of $\mathfrak{s}$ on the radical remain unchanged, while the brackets of the radical $\mathfrak{r}$ adopt the form

$$[X_{n+i}', X_{n+j}'] = \frac{1}{n} C_{n+i,n+j}^{n+k} X_{n+k}' \qquad 1 \leqslant i, j, k \leqslant m. \tag{19}$$

Therefore, for $n \to \infty$ we obtain

$$[X_{n+i}', X_{n+j}'] = 0.$$

This shows that the Lie algebra $\mathfrak{s} \overrightarrow{\oplus}_R m L_1$ is an Inönü–Wigner contraction of $\mathfrak{g}$ [23]. Using either the well-known fact that the number of Casimir operators of a contraction is at least that of the contracted algebra or the general formulae given in [24] for arbitrary types of invariants, we conclude that

$$\mathcal{N}(\mathfrak{g}) \leqslant \mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R m L_1) \tag{20}$$

and the assertion follows from proposition 2. □

We now analyse the block matrix (10) more closely, in order to extract additional information on the constitution of the Casimir operators in dependence on the representation R.

Proposition 4. *A perfect Lie algebra* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R (\deg R) L_1$ *contains* $\mathcal{M}(\mathfrak{g}) := \deg(R) - \operatorname{rank} \rho_R(\mathfrak{s})$ *Casimir operators which depend only on the variables associated with the* $\mathfrak{s}$-*module* $(\deg R) L_1$.

Proof. If $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+\deg R}\}$ is a basis of $\mathfrak{g}$ such that $\{X_1, \ldots, X_n\}$ spans $\mathfrak{s}$ and $\{X_{n+1}, \ldots, X_{n+\deg R}\}$ is a basis of the radical $(\deg R) L_1$, then from the system of PDEs (2) giving the invariants of $\mathfrak{g}$ we can extract the following subsystem,

$$\left\{ \hat{X}_i' F := -C_{i,j+n}^k x_k \frac{\partial}{\partial x_{n+j}} F = 0 \right\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant \deg R, k\geqslant n+1} \tag{21}$$

which can be rewritten as

$$\rho_R(\mathfrak{s}) \left(\frac{\partial}{\partial x_{n+j}} F \right)^T = 0; \tag{22}$$

(21) is the maximal subsystem of (2) containing only variables of the radical of $\mathfrak{g}$, and therefore any solution to (21) extends trivially to an invariant of $\mathfrak{g}$. The number of independent solutions of (21) is given by $\deg(R) - \operatorname{rank} \rho_R(\mathfrak{s})$ applying standard results of differential equations (system (21) can be seen as an independent system). Moreover, since the algebra $\mathfrak{g}$ is perfect, the solutions of (21) can be chosen as polynomials in the variables of $(\deg R)L_1$. □

This provides some interesting consequences concerning the invariants of semidirect products corresponding to irreducible representations of minimal degree (that is, either the standard or the adjoint representation). To this extent denote by R_0 the irreducible representation of a semisimple Lie algebra of minimal degree.

Corollary 4. *If* $\deg R_0 < \dim \mathfrak{s}$ *then any invariant of* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{R_0} (\deg R_0) L_1$ *depends on the variables of* $\mathfrak{s}$ *and* $(\deg R_0)L_1$.

The proof is straightforward, since the subsystem (21) determined by the action of $\mathfrak{s}$ on the radical has no solutions, as follows easily taking into account the matrix $\rho_{R_0}(\mathfrak{s})$ and the fact that $\deg R_0 < \dim(\mathfrak{s})$. Therefore any solution to (2) must involve variables corresponding to elements of $\mathfrak{s}$. It can also be seen that the Casimir operators of $\mathfrak{s}$ do not provide solutions of the system, which implies that any invariant is necessarily a solution of the subsystem:

$$-\rho_{R_0}(\mathfrak{s})^T \left(\frac{\partial}{\partial x_j} F\right)^T = 0. \tag{23}$$

This is mainly the reason why for the inhomogeneous Lie algebras the Casimir operators depend on almost all the variables of the algebra, since the low dimension of the representation is not sufficient to generate solutions of the system (2). As a consequence, representations of this type constitute the most difficult case to solve, since no easy applicable general pattern to determine their invariants seems to exist. The best known example of this pathology is the special affine Lie algebras $\mathfrak{sa}(n, \mathbb{R})$ [18].

Corollary 5. *For any simple Lie algebra* $\mathfrak{s}$ *the semidirect product* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{ad\mathfrak{s}} (\dim \mathfrak{s}) L_1$ *has exactly two* $\operatorname{rank}(\mathfrak{s})$ *independent Casimir operators. In particular, it has at least two quadratic Casimir operators.*

Proof. Let $A(\mathfrak{g})$ be the commutator matrix of $\mathfrak{g}$. The system (2) giving the invariants can be rewritten as

$$A(\mathfrak{g}) := \left(\begin{array}{c|c} A(\mathfrak{s}) & \rho_R(\mathfrak{s}) \\ \hline -\rho_R(\mathfrak{s})^t & 0 \end{array}\right) \left(\frac{\partial F}{\partial x_i}\right)^T = 0. \tag{24}$$

From proposition 4 we know that the subsystem

$$\rho_R(\mathfrak{s}) \left(\frac{\partial}{\partial x_{n+j}} F\right)^T = 0 \tag{25}$$

provides $\operatorname{rank}(\mathfrak{s})$ solutions of (22). Since the action of the Levi part $\mathfrak{s}$ on the Abelian radical is the adjoint action, it follows easily that there do not exist invariants of $\mathfrak{g}$ depending only on variables associated with $\mathfrak{s}$. This implies that the system (24) can be simplified to the following:

$$\left(\begin{array}{c|c} 0 & \rho_R(\mathfrak{s}) \\ \hline -\rho_R(\mathfrak{s})^T & 0 \end{array}\right) \left(\frac{\partial F}{\partial x_i}\right)^T = 0. \tag{26}$$

This shows that the algebra has exactly two $\operatorname{rank}(\mathfrak{s})$ independent invariants, which can be chosen as polynomials by the perfectness of the algebra. This proves the first assertion.

Suppose that $\{X_1, \ldots, X_n\}$ spans $\mathfrak{s}$ and let $C(2) = a^{ij} x_i x_j$ $(1 \leqslant i; j \leqslant n)$ be a quadratic Casimir operator of $\mathfrak{s}$. From (24) it is immediate that $C'(2) := a^{ij} x_{i+n} x_{j+n}$ is an invariant of $\mathfrak{g}$, since it is a solution of (24). From the matricial expression of (26) it also follows that $C(2)$ can be used to obtain a degree 2 solution of the system,

$$-\rho_R(\mathfrak{s})^T \left(\frac{\partial}{\partial x_j} F \right)^T = 0 \qquad (27)$$

where $1 \leqslant j \leqslant n$. Indeed, the polynomial

$$C''(2) := a^{ij} (x_{i+n} x_j + x_i x_{n+j}) \qquad (28)$$

is a solution of (26) and (27), thus an invariant of $\mathfrak{g}$. So for any degree 2 Casimir operator of $\mathfrak{s}$ we obtain two degree 2 operators of the perfect Lie algebra $\mathfrak{g}$. □

As a consequence of this result, it suffices to know the Casimir operators of a semisimple Lie algebra $\mathfrak{s}$ to determine all invariants of the corresponding perfect Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{ad\mathfrak{s}} (\dim \mathfrak{s}) L_1$, without having to solve the corresponding system (2) explicitly. The following example illustrates the procedure of the proof. Consider the Lorentz algebra $\mathfrak{so}(3, 1)$. Over the basis $\{J_i, K_i\}_{i=1,2,3}$ the brackets are given by

$$[J_i, J_j] = \varepsilon_{ijk} J_k \qquad [J_i, K_j] = \varepsilon_{ijk} K_k \qquad [K_i, K_j] = -\varepsilon_{ijk} J_k. \qquad (29)$$

The Casimir operators are

$$\begin{aligned} C_1 &= J_1^2 + J_2^2 + J_3^2 - K_1^2 - K_2^2 - K_3^2 \\ C_2 &= J_1 K_1 + J_2 K_2 + J_3 K_3. \end{aligned} \qquad (30)$$

Consider the Lie algebra $\mathfrak{so}(3, 1) \overrightarrow{\oplus}_{ad\mathfrak{so}(3,1)} 6L_1$ over the basis $\{J_i, K_i, J_i', K_i'\}_{i=1,2,3}$, where the brackets are those of (29) joint with the following:

$$\begin{aligned} [J_i, J_j'] &= \varepsilon_{ijk} J_k' \qquad [J_i, K_j'] = \varepsilon_{ijk} K_k' \\ [K_i, J_j'] &= \varepsilon_{ijk} K'_k \qquad [K_i, K_j'] = -\varepsilon_{ijk} J_k'. \end{aligned} \qquad (31)$$

It is elementary to verify that the rank of the commutator matrix is 8, so that the algebra has four invariants. The perfectness ensures that they can be chosen as polynomials. Using formula (28) applied to the invariants (30), we obtain the following polynomials:

$$\begin{aligned} C_1' &= J_1'^2 + J_2'^2 + J_3'^2 - K_1'^2 - K_2'^2 - K_3'^2 \\ C_2' &= J_1' K_1' + J_2' K_2' + J_3' K_3' \\ C_3' &= J_1 J_1' + J_2 J_2' + J_3 J_3' - K_1 K_1' - K_2 K_2' - K_3 K_3' \\ C_4' &= J_1 K_1' + J_2 K_2' + J_3 K_3' + J_1' K_1 + J_2' K_2 + J_3' K_3 \end{aligned} \qquad (32)$$

Either by direct verification or by analysis of the corresponding system (2), these functions are invariants of the algebra and constitute a fundamental set of invariants of $\mathfrak{so}(3, 1) \overrightarrow{\oplus}_{ad\mathfrak{so}(3,1)} 6L_1$.

In view of the preceding examples, we can consider the adjoint representation as a limiting case, since it allows an explicit determination of the invariants of the algebra $\mathfrak{s} \overrightarrow{\oplus}_{ad\mathfrak{s}} (\dim \mathfrak{s}) L_1$, while for lower dimensional representations the invariants of $\mathfrak{s}$ are of no use to determine the Casimir operators of the corresponding algebra. One can therefore expect that for representations R whose degree exceeds the dimension of $\mathfrak{s}$, all invariants will be obtained by applying proposition 4. With some restrictions, this will be indeed the general pattern.

Proposition 5. *For sufficient high degree* $\deg R$*, the perfect Lie algebra* $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ *satisfies the following equality:*

$$\mathcal{N}(\mathfrak{g}) = \mathcal{M}(\mathfrak{g}).$$

The proof is a direct consequence of the procedure used in the proof of proposition 4 and the degree of the representation R. Two important remarks must however be made. In general $\deg R > \dim\mathfrak{s}$ does not automatically imply that the invariants of $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ are independent of the variables $\{x_1, \ldots, x_n\}$ associated with $\mathfrak{s}$. The most elementary example showing this is the Lie algebra $\mathfrak{su}(3)\overrightarrow{\oplus}_{6\oplus\bar{3}}9L_1$, where $6 = \mathrm{Sym}^2 3$, 3 denoting the three-dimensional standard (quark) representation and $\bar{3}$ the dual antiquark representation. This algebra has three Casimir operators, but here $\mathcal{M}(\mathfrak{su}(3)\overrightarrow{\oplus}_{6\oplus\bar{3}}9L_1) = 1$, so that in spite of having a representation whose degree exceeds the dimension of $\mathfrak{su}(3)$, two Casimir operators depend essentially on the variables of the Levi part. This is due to the combination of a representation and a dual representation, which will reduce the number of solutions found from the subsystem determined by the matrix $\rho_R(\mathfrak{su}(3))$. For other semisimple Lie algebras a similar pattern is observed for combinations of this type whose degree exceeds the dimension of the algebra in only some units. However, for sufficiently high dimension of the representation this anomaly will not occur any more, so that proposition 4 holds. In particular, for rank 1 simple Lie algebras the proposition holds for any representation R of degree $\geqslant 4$ [8].

The second important fact concerns the perfectness of the algebra. If we allow R to contain a copy of the trivial representation D_0, the preceding assertion is false. Again the Lie algebra $\mathfrak{su}(3)$ provides the example. Consider the Lie algebra $\mathfrak{su}(3)\overrightarrow{\oplus}_{3\oplus3\oplus3\oplus1}\mathfrak{r}$, where R is the representation $3\oplus3\oplus3\oplus1$ (1 denoting D_0) and $\mathfrak{r}$ is defined by the brackets

$$[X_{18}, X_i] = X_i \qquad 9 \leqslant i \leqslant 17. \tag{33}$$

An application of proposition 4 would provide two functions depending only on $\{X_9, \ldots, X_{17}\}$ ($\{X_1, \ldots, X_8\}$ being a basis of $\mathfrak{su}(3)$), but it can easily be verified that $\mathcal{N}(\mathfrak{su}(3)\overrightarrow{\oplus}_{3\oplus3\oplus3\oplus1}\mathfrak{r}) = 0$ holds. Therefore the absence of D_0 is an essential condition for proposition 4 to be applied.

5. Lower bound for $\mathcal{N}(\mathfrak{g})$ and Abelian radicals

In the preceding section, we have seen that for sufficiently high dimension of R the invariants of the Lie algebra $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ are completely determined by the subsystem (25). This fact leads us naturally to ask if the decomposition of R into irreducible representations allows us to find invariants of subalgebras which extend naturally to invariants of the total algebra. This method has already been applied to algebras having a rank 1 Levi part [8], where proposition 4 holds generically because all representations are self-dual. Now, for higher ranks, we have already seen that, even if $\deg R > \dim\mathfrak{s}$, the subsystem (25) does not automatically provide all invariants of $\mathfrak{g}$, if the difference $\deg R - \dim\mathfrak{s}$ is small. Thus, before trying to generalize the results of [8], we must guarantee that for the representations R used proposition 4 holds. We can ask which is, for a given semisimple Lie algebra $\mathfrak{s}$, the minimal positive integer such that if R is an irreducible representation of $\mathfrak{s}$ of degree $\deg R := \dim(\mathfrak{s}) + n(\mathfrak{s})$, then all Casimir operators of $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ are solutions of subsystem (25). This integer $n(\mathfrak{s})$ will be called the normalization index of $\mathfrak{s}$. As follows from table 1, for A_2 we have $n(A_2) = 2$, while for rank 1 algebras the index equals 1. Now let $R = \sum_i R_i$ be the decomposition of R into irreducible representations of $\mathfrak{s}$.

Proposition 6. *Let* $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ *be a perfect Lie algebra. If* $\deg R_i - \dim(\mathfrak{s}) \geqslant n(\mathfrak{s})$ *for all i and* $R_i = R_j$ *if and only if* $i = j$, *then*

$$\sum_i \mathcal{N}(\mathfrak{g}_i) < \mathcal{N}(\mathfrak{g}) \tag{34}$$

where $\mathfrak{g}_i := \mathfrak{s}\overrightarrow{\oplus}_{R_i}(\deg R_i)L_1$.

Proof. Since the degree of R exceeds the normalization index of $\mathfrak{s}$, proposition 4 holds and any invariant of $\mathfrak{g}$ depends only on the variables associated with the radical $(\deg R)L_1$. Moreover, since for any irreducible component the same property is preserved, we have that

$$\mathcal{N}(\mathfrak{g}_i) = \mathcal{M}(\mathfrak{g}_i) \quad \forall i \tag{35}$$

and any Casimir invariant C_i of $\mathfrak{g}_i$ extends to an invariant of $\mathfrak{g}$ naturally. Since the irreducible components of R are supposed to be pairwise in equivalent, this ensures that the set $\bigcup_i\{\mathfrak{F}_i\}$ is formed by independent functions, where $\mathfrak{F}_i$ denotes a fundamental system of invariants of $\mathfrak{g}_i$. Since

$$\mathcal{N}(\mathfrak{g}) = \mathcal{M}(\mathfrak{g}) = \deg R - \dim\mathfrak{s} > \sum_i(\deg R_i - \dim\mathfrak{s}) \tag{36}$$

the inequality (34) is strict. □

Formula (34) gives the maximal number of invariants that can be deduced taking into account the decomposition of R into irreducible representations. Here the nonequivalence of the components is important, since otherwise some of the elements in $\bigcup_i\{\mathfrak{F}_i\}$ may be dependent. If we introduce an additional quantity η, which is defined as the smallest integer such that a subset $\mathfrak{F}' \subset \bigcup_i\{\mathfrak{F}_i\}$ is formed by independent functions and such that $|\mathfrak{F}'| = |\bigcup_i\{\mathfrak{F}_i\}| - \eta$ (that is, η gives the minimal number of invariants which must be removed to obtain an independent set), then we can enounce a generalization of the preceding result:

Corollary 6. *Let* $\mathfrak{s}\overrightarrow{\oplus}_R(\deg R)L_1$ *be a perfect Lie algebra. If* $\deg R_i - \dim(\mathfrak{s}) \geqslant n(\mathfrak{s})$ *for all i, then*

$$\sum_i \mathcal{N}(\mathfrak{g}_i) - \eta < \mathcal{N}(\mathfrak{g}) \tag{37}$$

where $\mathfrak{g}_i := \mathfrak{s}\overrightarrow{\oplus}_{R_i}(\deg R_i)L_1$.

The essential condition of these lower bounds is that $\deg R_i \geqslant n(\mathfrak{s})$ holds, as can easily be seen taking the following algebra: $\mathfrak{g} = \mathfrak{su}(3)\overrightarrow{\oplus}_{3\oplus3\oplus3}9L_1$. From table 1 we see that $\mathcal{N}(\mathfrak{g}) = 1$, and applying proposition 4 the invariant is easily found to be

$$(x_9x_{13}x_{17} - x_9x_{14}x_{16} - x_{11}x_{13}x_{15} - x_{10}x_{12}x_{17} + x_{11}x_{12}x_{16} + x_{10}x_{14}x_{15}). \tag{38}$$

Considering the decomposition of the representation we have $\mathfrak{g}_i \simeq \mathfrak{su}(3)\overrightarrow{\oplus}_3 3L_1$. This shows that formula (34) does not hold for this case, but even more, that no invariant of $\mathfrak{g}_i$ extends to an invariant of $\mathfrak{g}$, because the invariants of $\mathfrak{g}_i$ depend on the variables associated with the Levi part $\mathfrak{su}(3)$.

6. Conclusions

We have completed the classical result of [11] by showing that a perfect Lie algebra $\mathfrak{g}$ always has nontrivial invariants for the coadjoint representation, which is not an obvious fact in view of Lie algebras like the special affine $\mathfrak{sa}(n, \mathbb{R})$, which have only invariants independently of the rank. This result follows from the special structure that the radical $\mathfrak{r}$ of a perfect Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ must have, excluding the existence of elements in $\mathfrak{r}$ which act as derivations on the maximal nilpotent ideal of $\mathfrak{g}$ (i.e., of total elements in $\mathfrak{r}$). This shows that the existence of a fundamental set of invariants formed by Casimir operators follows from the perfectness of the algebra, and that additional assumptions such as the existence of a nondegenerate bilinear form (as in the semisimple case) have an effect on the degree of these operators. For nonsemisimple perfect algebras there are no general formulae giving the number of independent Casimir invariants, but useful upper bounds for $\mathcal{N}(\mathfrak{g})$ can be established analysing the structure of the representation R of the Levi part expressing the Levi decomposition. Indeed the case of Abelian radicals suffices, since the general case of arbitrary radicals reduces to the latter by means of IW-contractions. Starting from the upper bound obtained, we can deduce general results concerning the structure of the Casimir operators, specifically on the variables they depend on. In particular, it is shown that the number and structure of the invariants of Lie algebras $\mathfrak{s} \overrightarrow{\oplus}_{ad(\mathfrak{s})} \dim \mathfrak{s} L_1$ are completely determined by the Casimir operators of the semisimple part $\mathfrak{s}$. Moreover, for high-dimensional representation the invariants will be independent of the variables associated with the Levi. Therefore the invariants are comprised in certain subsystems of (2). An interesting consequence of this is the possibility of obtaining invariants from perfect subalgebras associated with the decomposition of R into irreducible representations. This result is of interest for branching rules, since it allows us to determine those invariants (and in consequence those states) which depend essentially on the whole representation, and not on their irreducible components. This also relates to the problem of missing label operators [25], where the subalgebras to be considered are those corresponding to the different irreducible components of R, and representations involving tensor products. The physical application is worthy, since it allows important reductions. For example, as is well known, boson-like pairs ZN of neutrons and protons (outside the closest closed shell) uniquely define a representation $R = (\mathrm{ZN}, 0, 0, 0, 0)$ of A_5, which decomposes as a direct sum of A_2-representations according to the Arima–Iachello formula [26], and allow a classification of levels of rotational-like nuclei by means of the weight diagrams of $\mathfrak{su}(3)$ [27]. Now the invariants of the Lie algebra $A_5 \overrightarrow{\oplus}_R (\deg R) L_1$ can be described in terms of the decomposition of R as a sum of $\mathfrak{su}(3)$-representations as soon as $\mathrm{ZN} \geqslant 4$ ($\mathrm{ZN} \neq 6, 9$, where a copy of the trivial representation of $\mathfrak{su}(3)$ appears), since we would have $\deg R > n(A_5) + \dim(A_5) \geqslant n(A_2) + \dim(A_2)$, which ensures that the semisimple part does not intervene in the solutions of the system (2).

Another potential application of the results in section 5 arises in the theory of similarity solutions of differential equations [28]. Indeed, if $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R (\deg R) L_1$ is the symmetry algebra of the system

$$F_v \left(x_i, u_j, \frac{\partial u_j}{\partial x_i}, \ldots, \frac{\partial^r u_j}{\partial x_{i_1} \ldots \partial x_{i_q}} \right) = 0 \tag{39}$$

with ($v = 1, \ldots, l$; $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$) and $R = \sum_i R_i$ is the decomposition of R, then the subalgebras $\mathfrak{g}_i = \mathfrak{s} \overrightarrow{\oplus}_{R_i} (\deg R_i) L_1$ can be used to obtain similarity solutions of (39), with the additional fact that the invariants of $\mathfrak{g}_i$ used for the reduction of the system are indeed invariants of the total (symmetry) algebra.

An important question that arises is whether the normalization index $n(\mathfrak{s})$ of a semisimple Lie algebra can be determined as a function of the rank of the Lie algebra, or if further quantities have to be considered. An explicit formula for this index would provide us with a quite systematic method to determine the invariants of perfect Lie algebras. In any case it would allow us to distinguish the representations R which must be analysed separately.

Of particular interest are the algebras $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R (\deg R) L_1$ whose defining representations satisfy $\deg R - \dim(\mathfrak{s}) < n(\mathfrak{s})$. Inhomogeneous Lie algebras constitute a special case of this, and the difficulty of obtaining their invariants is well known [6]. It would be desirable to find some method in order to predict the appearance of the Casimir operators of these algebras. This reduces to finding general criteria to solve systems like (23), where the traditional methods are not always applicable, due to the division of the variables intervening in the differential operators ∂_{x_j} and those defining the coefficients of the system.

Finally, general criteria to describe the invariants of an algebra which depend on the variables of the representations R and not on the variables of some of its components would also provide valuable information concerning the physical quantities described and codified by these representations. This particularly concerns the representations of exceptional Lie algebras, whose branching rules and tensor product decompositions have shown some importance in the problem of state classification [29, 30], and which constitute nowadays a common tool in high energy physics, particularly in string theory [31].

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